Proceedings of the Jangjeon Mathematical Society
28 (2025), No. 4, pp. 423 - 431

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2025.28.4.004

NEUTROSOPHICATION OF POSITIVE IMPLICATIVE AND ASSOCIATIVE FILTERS IN BASIC LOGIC ALGEBRAS

A. IBRAHIM, S. KARUNYA HELEN GUNASEELI,
AND FLORENTIN SMARANDACHE

Abstract. The study of neutrosophy has grown exponentially in recent years. The key goal of this article is to explore the idea of neutrosophic positive implicative and associative filters in Basic Logic (BL) algebras with suitable illustrations. Also, we prove that every neutrosophic positive implicative and neutrosophic associative filter is a neutrosophic filter in BL-algebras. Also, we explore a set of equivalent conditions. Finally, we establish the relationship between the neutrosophication of positive implicative, associative, and fantastic filters.

2020 Mathematics Subject Classification. 03G25, 03E72, 03F55,06F35

Keywords and phrases. BL-algebra, filter, neutrosophic filter, neutrosophic positive implicative filter, neutrosophic associative filter.

1. INTRODUCTION

Hajek [2] developed Basic Logic (BL) algebras to examine various valued forms of logic through algebraic methods. He was driven by two different types of reasons to instigate BL-algebras. The major was supplying an analytical equivalent known as Basic Logic [11,3]. The next objective was to develop an analytical mean. One of the foundations upon which the model of BL-algebras was built was the philosophy of MV-algebras. On the one hand, it can help us understand what is feasible, while on the other, it can also set certain boundaries.

Florentin Smarandache's [10] philosophy of neutrosophy addresses the ambiguities and imprecision of data. The foundations of the study of neutrosophical logic, probability, sets, and statistics are all provided by this discipline. Information system applications, IT, decision support systems, etc. have all included this principle in many studies. Jun [4] et al. instigated the concepts of positive implicative and associative filters in lattice implication algebras [12]. Liu and Li [5] instigated the concept of fuzzy positive implicative filters of BL-algebras. Recently, the authors studied the neutrosophication of fantastic implicative and n-fold implicative filters [6,7,8] in BL-algebras.

Our major contributions:

The ideas of neutrosophic positive implicative and associative filters are applied in BL-algebras. We obtain a few equivalent requirements for a neutrosophic filter to be neutrosophic positive implicative, and associative. Finally, we establish the relationship among the neutrosophic positive implicative, associative and fantastic filters in BL-algebras.

2. PRELIMINARIES

In this part, few of the definitions and findings from the literature are referred to progress the major conclusions.

Definition 2.1 [11, 5] A BL-algebra $(\mathcal{G}, \wedge, \vee, \circ, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) such that the subsequent requirements are persuaded for all $\sigma, \tau, \varphi \in \mathcal{G}$,

(i) $(\mathcal{G}, \wedge, \vee, \circ, \rightarrow, 0, 1)$ is a bounded lattice,
(ii) $(\mathcal{G}, \circ, 1)$ is a commutative monoid,
(iii) $'\circ'$ and $'\rightarrow'$ form an adjoint pair, that is, $\varphi \leq \sigma \rightarrow \tau$ if and only if $\sigma \circ \varphi \leq \tau$ for all $\sigma, \tau, \varphi \in \mathcal{G}$,
(iv)$\sigma \wedge \tau = \sigma \circ (\sigma \rightarrow \tau)$,
(v) $(\sigma \rightarrow \tau) \vee (\tau \rightarrow \sigma) = 1$.

Proposition 2.2[11,5] The succeeding requirements are persuaded in a BL- algebra $\mathcal{G}$ for all $\sigma, \tau, \varphi \in \mathcal{G}$,
(i) $\tau \rightarrow (\sigma \rightarrow \varphi) = \sigma \rightarrow (\tau \rightarrow \varphi) = (\sigma \circ \tau) \rightarrow \varphi$,
(ii) $1 \rightarrow \sigma = \sigma$,
(iii) $\sigma \leq \tau$ if and only if $\sigma \rightarrow \tau = 1$,
(iv) $\sigma \vee \tau = ((\sigma \rightarrow \tau) \rightarrow \tau) \wedge ((\tau \rightarrow \sigma) \rightarrow \sigma)$,
(v) $\sigma \leq \tau$ implies $\tau \rightarrow \varphi \leq \sigma \rightarrow \varphi$,
(vi) $\sigma \leq \tau$ implies $\varphi \rightarrow \sigma \leq \varphi \rightarrow \tau$,
(vii) $\sigma \rightarrow \tau \leq (\varphi \rightarrow \sigma) \rightarrow (\varphi \rightarrow \tau)$,
(viii) $\sigma \rightarrow \tau \leq (\tau \rightarrow \varphi) \rightarrow (\tau \rightarrow \varphi)$,
(ix) $\sigma \leq (\sigma \rightarrow \tau) \rightarrow \tau$,
(x) $\sigma \circ (\sigma \rightarrow \tau) = \sigma \wedge \tau$,
(xi) $\sigma \circ \tau \leq \sigma \wedge \tau$,
(xii) $\sigma \rightarrow \tau \leq (\sigma \circ \varphi) \rightarrow (\tau \circ \varphi)$,
(xiii) $\sigma \circ (\tau \rightarrow \varphi) \leq \tau \rightarrow (\sigma \circ \varphi)$,
(xiv) $(\sigma \rightarrow \tau) \circ (\tau \rightarrow \varphi) \leq \sigma \rightarrow \varphi$,
(xv) $(\sigma \circ \sigma^*) = 0$.

Definition 2.3[1,9] A neutrosophic subset $\mathcal{C}$ of the universe $\mathcal{U}$ is a triple (T_C, I_C, F_C) where $T_C : \mathcal{U} \rightarrow [0,1]$ $I_C : \mathcal{U} \rightarrow [0,1]$ and $F_C : \mathcal{U} \rightarrow [0,1]$ represents truth membership, indeterminacy and false membership functions respectively where $0 \leq T_C(\sigma) + I_C(\sigma) + F_C(\sigma) \leq 3$ for all $\sigma \in \mathcal{U}$.

Definition 2.4[8] A neutrosophic set $\mathcal{C}$ of a BL- algebra $\mathcal{G}$ is called a neutrosophic filter, if it persuades the requirements:
(i) $T_C(\sigma) \leq T_C(1), I_C(\sigma) \geq I_C(1)$ and $F_C(\sigma) \geq F_C(1)$,
(ii) $min\{T_C(\sigma \rightarrow \tau), T_C(\sigma)\} \leq T_C(\tau), min\{I_C(\sigma \rightarrow \tau), I_C(\sigma)\} \geq I_C(\tau)$ and $min\{F_C(\sigma \rightarrow \tau), F_C(\sigma)\} \geq F_C(\tau)$ for all $\sigma, \tau \in \mathcal{G}$.

Proposition 2.5[6] Let $\mathcal{C}$ be a neutrosophic filter of $\mathcal{G}$ if and only if
(i) If $\sigma \leq \tau$ then $T_C(\sigma) \leq T_C(\tau), I_C(\sigma) \geq I_C(\tau)$ and $F_C(\sigma) \geq F_C(\tau)$,
(ii) $T_C(\sigma \circ \tau) \geq min\{T_C(\sigma), T_C(\tau)\}, I_C(\sigma \circ \tau) \leq min\{I_C(\sigma), I_C(\tau)\}$ and $F_C(\sigma \circ \tau) \leq min\{F_C(\sigma), F_C(\tau)\}$for all $\sigma, \tau \in \mathcal{G}$.

Proposition 2.6 [6, 7] Let $\mathcal{C}$ be a neutrosophic filter of $\mathcal{G}$ for all $\sigma, \tau, \varphi \in \mathcal{G}$ then the following hold.
(i) $T_C(\sigma \to \tau) = T_C(1), then\ T_C(\sigma)_C(\tau), I_C(\sigma \to \tau) = I_C(1),$
$then\ I_C(\sigma) \geq I_C(\tau), F_C(\sigma \to \tau) = F_C(1), then\ F_C(\sigma) \geq F_C(\tau)$
$(ii) T_C(\sigma \wedge \tau) = min\{T_C(\sigma), T_C(\tau)\}, I_C(\sigma \wedge \tau) = min\{I_C(\sigma), I_C(\tau)\},$
$F_C(\sigma \wedge \tau) = min\{F_C(\sigma), F_C(\tau)\}$
$(iii) T_C(\sigma \circ \tau) = min\{T_C(\sigma), T_C(\tau)\}, I_C(\sigma \circ \tau) = min\{I_C(\sigma), I_C(\tau)\},$
$F_C(\sigma \circ \tau) = min\{F_C(\sigma), F_C(\tau)\}$
$(iv) T_C(0) = min\{T_C(\sigma), T_C(\sigma^*)\}, I_C(0) = min\{I_C(\sigma), I_C(\sigma^*)\},$
$F_C(0) = min\{F_C(\sigma), F_C(\sigma^*)\}$

Definition 2.7 [7] Let $\mathcal{C}$ be called a neutrosophic fantastic filter of $\mathcal{G}$, if it persuades the subsequent requirements for all $\sigma, \tau, \varphi \in \mathcal{G}$,
$(i) T_C(1) \geq T_C(\sigma), I_C(1) \leq I_C(\sigma), F_C(1) \leq F_C(\sigma).$
$(ii) min\{T_C(\sigma \to \tau), T_C(\sigma)\} \leq T_C(\tau), min\{I_C(\sigma \to \tau), I_C(\sigma)\} \geq I_C(\tau) and$
$min\{F_C(\sigma \to \tau), F_C(\sigma) \geq F_C(\tau).$
$(iii) T_C((\sigma \to \tau) \to \tau) \to \sigma) \geq min\{T_C(\varphi \to (\tau), \mathrm{T}_C(\varphi)\},$
$I_C((\sigma \to \tau) \to \tau) \to \sigma) \leq min\{I_C(\varphi \to (\tau \to \sigma)), I_C(\varphi)\},$
$F_C((\sigma \to \tau) \to \tau) \to \sigma) \leq min\{F_C(\varphi \to (\tau \to \sigma)), F_C(\varphi)\}.$

Proposition 2.8 [7] Let $\mathcal{C}$ be a neutrosophic fantastic filter of $\mathcal{G}$ if and only if $T_C(((\sigma \to \tau) \to \tau) \to \sigma) \geq T_C(\tau \to \sigma), I_C(((\sigma \to \tau) \to) \to \sigma) \leq I_C(\tau \to \sigma)$ and $F_C(((\sigma \to \tau) \to \tau) \to \sigma) \leq F_C(\tau \to \sigma)$ for all $\sigma, \tau \in \mathcal{G}$.

Definition 2.9 [5] A fuzzy set μ of $\mathcal{G}$ is called a fuzzy positive implicative filter of $\mathcal{G}$, if it satisfies, (i) $\mu(1) \geq \mu(\sigma)$
$(ii) \mu(\tau) \geq min\{\mu(\sigma \to ((\tau \to \varphi) \to \tau)), \mu(\sigma)$ for all $\sigma, \tau \in \mathcal{G}$.

3. Neutrosophic positive implicative filter

Here, we put forward the conception of a neutrosophic positive implicative filter and confer its features with illustrations.

Definition 3.1 Let $\mathcal{C}$ be called a neutrosophic positive implicative filter of a BL-algebra $\mathcal{G}$ if it persuades,
(i) $T_C(\sigma) \leq T_C(1), I_C(\sigma) \geq I_C(1)$and $F_C(\sigma) \geq F_C(1)$
(ii) $min\{T_C(\sigma \to \tau), T_C(\sigma)\} \leq T_C(\tau), min\{I_C(\sigma \to \tau), I_C(\sigma)\} \geq I_C(\tau)$ and $min\{F_C(\sigma \to \tau), F_C(\sigma)\} \geq F_C(\tau)$
(iii)$min\{T_C(\sigma \to ((\tau \to \varphi) \to \tau), T_C(\sigma)\} \leq T_C(\sigma),$
$min\{I_C(\sigma \to ((\tau \to \varphi) \to \tau), I_C(\sigma)\} \geq I_C(\sigma),$
$min\{F_C(\sigma \to (\tau \to \varphi) \to \tau), F_C(\sigma)\} \geq F_C(\sigma)$ for all $\sigma, \tau, \varphi \in \mathcal{G}$.

Example 3.2 Let $\mathcal{C} = \{0, \alpha_{11}, \beta_{11}, \gamma_{11}, 1\}$.The bi-fold operations are given by the subsequent tables (1) and (2).

Consider
$\mathcal{C} = (0, [0.4, 0.6, 0.6]), (\alpha_{11}, [0.5, 0.6, 0.6]), (\beta_{11}, [0.5, 0.6, 0.6]), (\gamma_{11}, [0.5, 0.6, 0.6]), (1, [0.6, 0.6, 0.6]).$

Table 1. $'\circ'$ Operation

$\circ$	0	α_{11}	β_{11}	γ_{11}	1
0	1	1	1	1	1
α_{11}	γ_{11}	1	1	1	1
β_{11}	β_{11}	γ_{11}	1	1	1
γ_{11}	α_{11}	β_{11}	γ_{11}	1	1
1	0	α_{11}	β_{11}	γ_{11}	1

Table 2. $'\rightarrow'$ Operation

$\rightarrow$	0	α_{11}	β_{11}	γ_{11}	1
0	1	1	1	1	1
α_{11}	α_{11}	1	1	1	1
β_{11}	0	α_{11}	1	1	1
γ_{11}	0	α_{11}	β_{11}	1	1
1	0	α_{11}	β_{11}	γ_{11}	1

It is evident that $\mathcal{C}$ assures the definition 3.1.
Hence, $\mathcal{C}$ is a neutrosophic positive implicative filter of $\mathcal{C}$.

Proposition 3.3 Every neutrosophic positive implicative filter is a neutrosophic filter. But the adverse may not be true.
Proof: Let $\mathcal{C}$ be a neutrosophic positive implicative filter of $\mathcal{G}$.
To prove: $\mathcal{C}$ is a neutrosophic filter of $\mathcal{G}$.
Then, putting $\varphi = \tau$ in (ii) of the definition 3.1, we get
$T_C(\tau) \geq min\{T_C((\sigma \rightarrow (\tau \rightarrow \tau) \rightarrow \tau), T_C(\sigma)$
$T_C(\tau) \geq min\{T_C(\sigma \rightarrow (1 \rightarrow \tau), T_C(\sigma)\}$
$T_C(\tau) \geq min\{T_C(\sigma \rightarrow \tau), T_C(\sigma)\}$
Similarly, we can prove for I_C, F_C.
Therefore, by the definition 2.4, $\mathcal{C}$ is a neutrosophic filter of $\mathcal{G}$.
The adverse part can be proved by an example.

Example 3.4 Let $\mathcal{D} = \{0, \alpha_{11}, \beta_{11}, \gamma_{11}, 1\}$. The bi-fold operations are specified by the tables (1) and (2).
Let $\mathcal{D} =$
$(0, [0.6, 0.4, 0.4]), (\alpha_{11}, [0.6, 0.4, 0.4]), (\beta_{11}, [0.6, 0.4, 0.4]), (\gamma_{11}, [0.6, 0.4, 0.4]), (1, [0.8, 0.3, 0.3])$
be a neutrosophic filter of $\mathcal{G}$.
Here, $\mathcal{D}$ is not a neutrosophic positive implicative filter.
Since,$T_D(\beta_{11}) = 0.4 \ngeq 0.5 = T_D(\alpha_{11})$.

Proposition 3.5 Let $\mathcal{C}$ be a neutrosophic positive implicative filter of $\mathcal{G}$ if and only if it persuades $T_C(\sigma) \geq T_C((\sigma \rightarrow \tau) \rightarrow \sigma), I_C(\sigma) \leq I_C((\sigma \rightarrow \tau) \rightarrow \sigma), F_C(\sigma) \leq F_C((\sigma \rightarrow \tau) \rightarrow \sigma)$ for all $\sigma, \tau \in \mathcal{G}$.

Proof: Let $\mathcal{C}$ be a neutrosophic positive implicative filter of $\mathcal{G}$.
Then $T_C(\sigma) \leq min\{T_C(1 \rightarrow (\sigma \rightarrow \tau) \rightarrow \sigma), T_C(1)\}$
$= min\{T_C((\sigma \rightarrow \tau) \rightarrow \sigma), T_C(1)\} = T_C((\sigma \rightarrow \tau) \rightarrow \sigma)$

Therefore, $T_C(\sigma) \geq T_C((\sigma \to \tau) \to \sigma)$ for all $\sigma, \tau \in \mathcal{G}$.
Similarly, we can prove for I_C, F_C.
Conversely, suppose that $T_C(\sigma) \geq T_C((\sigma \to \tau) \to \sigma)$
Then, $T_C(\tau) \geq T_C((\tau \to \varphi) \to \tau)$ for all $\sigma, \tau, \varphi \in \mathcal{G}$,
$\geq min\{T_C(\sigma \to (\tau \to \varphi)), T_C(\sigma)\}$ [By the definition 2.4]
Similarly, we can prove for I_C, F_C.
Therefore, $\mathcal{C}$ is a neutrosophic positive implicative filter of $\mathcal{G}$.

Proposition 3.6 Every neutrosophic positive implicative filter of G is a neutrosophic fantastic filter.

Proof: Let $\mathcal{C}$ be a neutrosophic positive implicative filter of $\mathcal{G}$.
Then $\mathcal{C}$ is a neutrosophic filter of $\mathcal{G}$. [From the proposition 3.3]
Therefore, it is enough to prove $T_C(((\sigma \to \tau) \to \tau) \to \sigma) \geq T_C(\tau \to \sigma)$.
Consider the following,
Since, $\sigma \circ ((\sigma \to \tau) \to \tau) \leq \sigma$, we have $\sigma \leq ((\sigma \to \tau) \to \tau) \to \sigma$
From the proposition 2.2, we get $((\sigma \to \tau)\tau) \to \sigma \leq \sigma \to \tau$,
and also we have $((((\sigma \to \tau) \to \tau) \to \sigma) \to \tau) \to ((\sigma \to \tau) \to \tau) \to \sigma)$
$\geq (\sigma \to \tau) \to (((\sigma \to \tau) \to \tau) \to \sigma)$
$\geq ((\sigma \to \tau) \to \tau) \to ((\sigma \to \tau) \to \sigma) \geq \tau \to \sigma$
Since, $\tau \to \sigma \in \mathcal{C}$, we get $((((\sigma \to \tau) \to \tau) \to \sigma) \to \tau) \to ((\sigma \to \tau) \to \tau) \to \sigma) \in \mathcal{C}$.
Since, $\mathcal{C}$ is a neutrosophic positive implicative filter, $((\sigma \to \tau) \to \tau) \to \sigma \in \mathcal{C}$.
Therefore, $T_C(((\sigma \to \tau) \to \tau) \to \sigma) \geq T_C(\tau \to \sigma)$,
$I_C(((\sigma \to \tau) \to \tau) \to \sigma) \leq I_C(\tau \to \sigma)$
and $F_C(((\sigma \to \tau) \to \tau) \to \sigma) \leq F_C(\tau \to \sigma)$for all $\sigma, \tau \in \mathcal{G}$.
Hence, from the proposition 2.8, $\mathcal{C}$ is a neutrosophic fantastic filter.

4. Neutrosophic associative filter

Here, we put forward the conception of a neutrosophic associative filter and confer its features with illustrations.

Definition 4.1 Let $\mathcal{C}$ be called a neutrosophic associative filter of a BL-algebra $\mathcal{G}$ if it persuades the following,

(i) $T_C(\sigma) \leq T_C(1), I_C(\sigma) \geq I_C(1)$and $F_C(\sigma) \geq F_C(1)$,
(ii) $min\{T_C(\sigma \to \tau), T_C(\sigma)\} \leq T_C(\tau), min\{I_C(\sigma \to \tau), I_C(\sigma)\} \geq I_C(\tau)$ $andmin\{F_C(\sigma \to \tau), F_C(\sigma)\} \geq F_C(\tau)$,
(iii) $min\{T_C(\sigma \to (\tau \to \varphi)), T_C(\sigma \to \tau)\} \leq T_C(\varphi)$,
$min\{I_C(\sigma \to (\tau \to \varphi)), I_C(\sigma \to \varphi)\} \geq I_C(\varphi)$,
$min\{F_C(\sigma \to (\tau \to \varphi)), F_C(\sigma \to \tau)\} \geq F_C(\varphi)$ for all $\sigma, \tau, \varphi \in \mathcal{G}$.

Example 4.2 Let $\mathcal{C} = \{0, \iota_{11}, \kappa_{11}, \lambda_{11}, 1\}$. The bi-fold operations are given by the subsequent tables (3) and (4).

Consider a neutrosophic set $\mathcal{C}$ of $\mathcal{G}$:
$\mathcal{C} = \{(0, [0.7, 0.8, 0.8]), (\iota_{11}, [0.7, 0.8, 0.8]), (\kappa_{11}, [0.7, 0.8, 0.8]), (\lambda_{11}, [0.7, 08, 0.8]), (1, [0.8, 0.7, 0.7])\}$.

TABLE 3. $'\circ'$ Operation

$\circ$	0	ι_{11}	κ_{11}	λ_{11}	1
0	1	ι_{11}	0	0	0
ι_{11}	1	ι_{11}	ι_{11}	ι_{11}	ι_{11}
κ_{11}	0	ι_{11}	1	κ_{11}	κ_{11}
λ_{11}	0	ι_{11}	κ_{11}	1	λ_{11}
1	0	ι_{11}	κ_{11}	λ_{11}	1

TABLE 4. $' \rightarrow'$ Operation

$\rightarrow$	0	ι_{11}	κ_{11}	λ_{11}	1
0	1	1	1	κ_{11}	1
ι_{11}	1	1	κ_{11}	λ_{11}	κ_{11}
κ_{11}	1	ι_{11}	1	κ_{11}	ι_{11}
λ_{11}	1	ι_{11}	1	1	ι_{11}
1	1	ι_{11}	κ_{11}	λ_{11}	0

It is evident that $\mathcal{C}$ assures the definition 4.1.
Hence, $\mathcal{C}$ is a neutrosophic associative filter of $\mathcal{G}$.

Proposition 4.3 Every neutrosophic associative filter is a neutrosophic filter. But, the adverse may not be true.
Proof: Let $\mathcal{C}$ be a neutrosophic associative filter of $\mathcal{G}$.
To prove: $\mathcal{C}$ is a neutrosophic filter of $\mathcal{G}$.
Then, taking $\sigma = 1$ in (ii) of the definition 4.1, we get
$T_C(\varphi) \geq min\{T_C(1 \rightarrow \tau \rightarrow \varphi)), T_C(1 \rightarrow \tau)\}$
$T_C(\varphi) \geq min\{T_C(\tau \rightarrow \varphi), T_C(\tau)$for all $\tau, \varphi \in \mathcal{G}$.

Similarly, we can prove for I_C, F_C.
Therefore, by the definition 2.4, $\mathcal{C}$ is a neutrosophic filter of $\mathcal{G}$.
The adverse part can be proved by an example.

Example 4.4 Let $\mathcal{D} = \{0, \iota_{11}, \beta_{11}, \lambda_{11}, 1\}$. The bi-fold operations are specified by the tables (3) and (4).
Let $\mathcal{D} = (0, [0.7, 0.4, 0.4]), (\iota_{11}, [0.7, 0.4, 0.4]), (\kappa_{11}, [0.6, 0.4, 0.4]), (\lambda_{11}, [0.7, 0.4, 0.4]), (1, [0.8, 0.3, 0.3])$ be a neutrosophic filter of $\mathcal{G}$.
Here, $\mathcal{D}$ is not a neutrosophic associative filter.
Since,$T_D(\kappa_{11}) = 0.6 \geq 0.7 = T_D(\iota_{11})$.

Proposition 4.5 Every neutrosophic associative filter with respect to σ contains σ itself.
Proof: Let $\sigma = 0$, then it is trivial.
If $\sigma \neq 0$,
Let $\mathcal{G}$ be a neutrosophic associative filter with respect to σ.
Then, $T_C(\sigma) \geq \min\{T_C(\sigma \rightarrow (1 \rightarrow \sigma)),$
$T_C(\sigma \rightarrow 1)\}$
$T_C(\sigma) \geq \min\{T_C(\sigma \rightarrow \sigma), T_C(1)\}$

$T_C(\sigma) \geq \min\{T_C(1), T_C(1)\}$
Therefore, $T_C(\sigma) = T_C(1)$. [By the definition 4.1 $(i), T_C(\sigma) \leq T_C(1)$]
Similarly, $I_C(\sigma) = I_C(1), F_C(\sigma) = F_C(1)$.

Proposition 4.6 Let $\mathcal{C}$ be a neutrosophic associative filter of a BL-algebra $\mathcal{G}$ if and only if it persuades $T_C((\sigma \to \tau) \to \varphi) \geq T_C(\sigma \to (\tau \to \varphi))$,
$I_C((\sigma \to \tau) \to \varphi) \leq I_C(\sigma \to (\tau \to \varphi))$,
$F_C((\sigma \to \tau) \to \varphi) \leq F_C(\sigma \to (\tau \to \varphi))$ for all$\sigma, \tau, \varphi \in \mathcal{G}$. (4.1)
Proof: Let $\mathcal{C}$ be a neutrosophic filter of $\mathcal{G}$.
Let $T_C((\sigma \to \tau) \to \varphi) \geq T_C(\sigma \to (\tau \to \varphi))$.
$Then, T_C(\varphi) \geq min\{T_C((\sigma \to \tau) \to \varphi), T_C(\sigma \to \tau)\}$
$T_C(\varphi) \geq min\{T_C(\sigma \to (\tau \to \varphi)), T_C(\sigma \to \tau)\}$ for all $\sigma, \tau, \varphi \in \mathcal{G}$.

Similarly, we can prove for I_C, F_C.
Hence, $\mathcal{C}$ is a neutrosophic associative filter.
Conversely, Let $\mathcal{C}$ be a neutrosophic associative filter of $\mathcal{G}$.
Consider $(\sigma \to (\tau \to \varphi) \to ((\sigma \to \tau) \to \varphi))$
$= (\tau \to \varphi) \to (\sigma \to ((\sigma \to \tau) \to \varphi))$ [From (i) of the proposition 2.2]
$= (\tau \to \varphi) \to ((\sigma \to \tau) \to (\sigma \to \varphi))$ [From (i) of the proposition 2.2]
$= 1 \in \mathcal{C}$. [From (i) of the proposition 2.2 and (v) of the definition 2.1]
Further,$T_C((\sigma \to \tau) \to \varphi) \geq min T_C(\sigma \to ((\tau \to \varphi) \to ((\sigma \to \tau) \to \varphi)))$,
$\mathrm{T}_C(\sigma \to (\tau \to \varphi)) = min\{T_C(1), T_C(\sigma \to (\tau \to \varphi))\} = T_C(\sigma \to (\tau \to \varphi))$
Therefore, $T_C((\sigma \to \tau) \to \varphi) \geq T_C(\sigma \to (\tau \to \varphi))$.
Similarly,$I_C((\sigma \to \tau) \to \varphi) \leq I_C(\sigma \to (\tau \to \varphi))$,
$F_C((\sigma \to \tau) \to \varphi) \leq F_C(\sigma \to (\tau \to \varphi))$ for all $\sigma, \tau, \varphi \in \mathcal{G}$.

Proposition 4.7 Let $\mathcal{C}$ be a neutrosophic associative filter of a BL-algebra $\mathcal{G}$ if and only if it persuades $T_C(\tau) \geq T_C(\sigma \to (\sigma \to \tau)), I_C(\tau) \leq I_C(\sigma \to (\sigma \to \tau)), F_C(\tau) \leq F_C(\sigma \to (\sigma \to \tau))$ for all $\sigma, \tau, \varphi \in \mathcal{G}$ (4.2)
Proof: Let $\mathcal{C}$ be a neutrosophic filter of $\mathcal{G}$.
To prove that $\mathcal{C}$ is a neutrosophic associative filter of $\mathcal{G}$, it is sufficient to verify that (4.1) and (4.2) are equivalent.
Assume that (4.1) holds. Putting $\sigma = \tau$ in (4.1).
$T_C((\tau \to \tau) \to \varphi)) \geq T_C(\tau \to (\tau \to \varphi))$
$T_C(1 \to \varphi) \geq T_C(\tau \to (\tau \to \varphi)$
$T_C(\varphi) \geq T_C(\tau \to (\tau \to \varphi))$
Similarly,$I_C(\varphi) \leq I_C(\tau \to (\tau \to \varphi)), F_C(\varphi) \leq F_C(\tau \to (\tau \to \varphi))$.
Hence, (4.1) implies (4.2).
Conversely, if (4.2) holds.
The following can be considered.
$(\sigma \to (\tau \to \varphi) \to (\sigma \to (\sigma \to ((\sigma \to \tau) \to \varphi)))$
$= 1 \to ((\sigma(\tau \to \varphi)) \to (\sigma \to (\sigma \to ((\sigma \to \tau) \to \varphi))))$
$= ((\tau \to \varphi) \to (\sigma \to (\sigma \to ((\sigma \to \tau) \to \varphi))) \to ((\sigma \to (\tau \to \varphi)) \to (\sigma \to (\sigma \to ((\sigma \to \tau) \to \varphi))))$
$= (\sigma \to (\tau \to \varphi)) \to ((\tau \to \varphi) \to (\sigma \to (\sigma \to \tau) \to \varphi)))) \to (\sigma \to (\sigma \to ((\sigma \to \tau) \to \varphi)))$
$\geq (\sigma \to (\tau \to \varphi)) \to (\sigma \to (\tau \to \varphi)) = 1 \in \mathcal{C}$
As $(\sigma \to (\tau \to \varphi) \to (\sigma \to (\sigma \to ((\sigma \to \tau) \to \varphi))) \in$C,

$Let \mathrm{T}_C((\sigma \to \tau) \to \varphi) \geq T_C(\sigma \to ((\sigma \to \tau) \to \varphi)))$
$\geq min\{T_C(\sigma \to (\tau \to \varphi)) \to (\sigma(\sigma \to ((\sigma \to \tau) \to \varphi)))\},$
$T_C(\sigma \to (\tau \to \varphi)) = min\{T_C(1), T_C(\sigma \to (\tau \to \varphi))\}$
$= T_C(\sigma \to (\tau \to \varphi))$
Therefore,$T_C((\sigma \to \tau) \to \varphi) \geq T_C(\sigma \to (\tau \to \varphi))$.
$Similarly, I_C((\sigma \to \tau) \to \varphi) \leq I_C(\sigma \to (\tau \to \varphi)),$
$F_C((\sigma \to \tau) \to \varphi) \leq F_C(\sigma \to (\tau \to \varphi))$ for all $\sigma, \tau, \varphi \in \mathcal{G}$.
Hence, (4.2) (4.1).
As (4.1) and (4.2) are equivalent by the proposition 4.6, $\mathcal{C}$ is a neutrosophic associative filter.

Proposition 4.8 Every neutrosophic associative filter of $\mathcal{G}$ is a neutrosophic fantastic filter.
Proof: Let $\mathcal{C}$ be a neutrosophic associative filter of $\mathcal{G}$.
Then, by the proposition 4.3, $\mathcal{C}$ is a neutrosophic filter of $\mathcal{G}$.
Consider
$(\tau \to \sigma) \to (\sigma \to (\sigma \to ((\sigma \to \tau) \to \tau)\sigma))$
$=(\tau \to \sigma) \to (\sigma \to ((\sigma \to \tau) \to \tau) \to (\sigma \to \sigma))$
$= (\tau \to \sigma) \to (\sigma \to ((\sigma \to \tau) \to \tau)) \to 1$
$= (\tau \to \sigma) \to (\sigma \to 1)$
$= (\tau \to \sigma) \to 1 = 1 \in \mathcal{C}$
which implies that $(\tau \to \sigma) \to (\sigma \to (\sigma \to ((\sigma \to \tau) \to \tau) \to \sigma)) \in \mathcal{C}$.
Therefore, $T_C(((\sigma \to \tau) \to \tau) \to \sigma) \geq T_C(\sigma \to (\sigma \to (((\sigma \to \tau) \to \tau) \to \sigma))$
$= min\{T_C((\tau \to \sigma) \to (\sigma \to (\sigma \to (\sigma \to \tau) \to \tau) \to \sigma)), T_C(\tau \to \sigma)\}$
$= min\{_C(1), T_C(\tau \to \sigma)\}$
$= T_C(\tau \to \sigma)$
Therefore,$T_C(((\sigma \to \tau) \to \tau) \to \sigma) \geq T_C(\tau \to \sigma)$ for all $\sigma, \tau \in \mathcal{C}$.
Similarly, we can prove for I_C, F_C.
Hence by the proposition 2.8, $\mathcal{C}$ is a neutrosophic fantastic filter.

5. Conclusion

The study of lattice implication algebras benefits greatly from the application of filter theory. In BL-algebras, we have put forth the conception of a neutrosophic positive implicative and associative filters. We have also demonstrated the neutrosophic nature of every positive implicative and associative filters. The following are our major results: (i) Every neutrosophic positive implicative filter of $\mathcal{G}$ is a neutrosophic fantastic filter. (ii) Every neutrosophic associative and positive implicative filters is a neutrosophic filter. (iii) Every neutrosophic associative filter of $\mathcal{G}$ is a neutrosophic fantastic filter. In future study, both theoretical extensions and real-life applications in artificial intelligence and decision making systems are possible.

Here, the Figure 1 given below shows the relationship between neutrosophic positive implicative , associative filter, and fantastic filters.

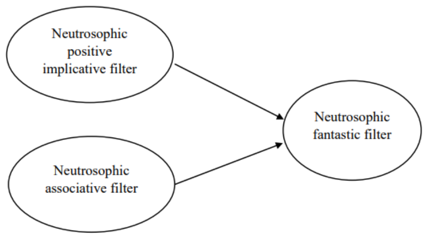

Figure 1. Relationship between the neutrosophic filters

References

[1] H.H. Abbass, Q.M.Luhaib, *On Smarandache Filter of a Smarandache BH-Algebra*, Journal of Physics (2019), 12–34.
[2] P.Hajek, *Metamathematics of fuzzy logic*, Klower Academic Publishers, Dodrecht (1999).
[3] M.Haveshki,A.Borumand Saied,E.Eslami, *Some types of filters in BL-algebras*, Soft Computing 10(2006), 657–664.
[4] Y.B.Jun,Y. Xu,and Qin, *Positive implicative and associative filter of lattice implication algebras* Bulletin of the Korean Mathematical Society , 35(1)(1998), 53-61.
[5] L.Z. Liu, and K. Li,*Fuzzy Boolean and positive implicative filters of BL-algebras*,Fuzzy Sets and Systems,152(2005), 333-348.
[6] A.Ibrahim, and S. Karunya Helen Gunaseeli, *On Neutrosophic filter of BL-algebras*,Ratio Mathematica, 47 (2023),141-150. .
[7] A.Ibrahim, and S. Karunya Helen Gunaseeli,*On Neutrosophic filter and fantastic filter of BL-algebras*, International Journal of Neutrosophic Science , 21(2)(2023), 59-67.
[8] A.Ibrahim,S. Karunya Helen Gunaseeli and Broumi Said, *Some Types of Neutrosophic Filters in Basic Logic Algebras*, Neutrosophic Systems With Applications, 11(2023),31–38.
[9] A.A Salama and H. Alagamy,*Neutrosophic Filters*, International Journal of Computer Science Engineering and Information Technology Research, 3(1) (2013), 307-312.
[10] Florentin Smarandache, *A Unifying field in Logics. Neutrosophy: Neutrosophic Probability*, Set and Logic, American Research Press, Rehoboth, USA, 1999.
[11] E.Turunen,*Boolean Deductive Systems of BL-algebras*, Archive for Mathematical Logic, 40(6)(2001), 467-473.
[12] Y. Xu,*Lattice implication algebras*, Journal of Southwest Jiaotong University,28(1) (1993),20-27.

Assistant Professor, P.G. and Research Department of Mathematics, H. H. The Rajah's College, Pudukkottai, Affiliated to Bharathidasan University, Trichirappalli, Tamilnadu, India.
Email address: dribrahimaadhil@gmail.com

Research Scholar, P.G. and Research Department of Mathematics, H. H. The Rajah's College, Pudukkottai, Affiliated to Bharathidasan University, Trichirappalli, Tamilnadu, India.
Email address: karjes821@gmail.com

Mathematics Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA.
Email address: smarand@unm.edu